# A GENERALIZATION OF MAXIMAL COHEN–MACAULAY APPROXIMATION THEOREM AND ITS APPLICATIONS

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Throughout, let $(R, \mathfrak{m}, k)$ be a noetherian local ring. By an *$R$-complex*, we mean a bounded complex of finitely generated $R$-modules.

## 1. A generalization of maximal Cohen–Macaulay approximation theorem

Auslander and Buchweitz ([1]) proved the following celebrated theorem, which is one of the most fundamental tools in commutative algebra.

**Theorem 1.1** (maximal Cohen–Macaulay approximation theorem)**.** *Let $R$ be a Cohen-Macaulay local ring with canonical module. For a finitely generated $R$-module $M$, there is a short exact sequence*

$$0 \to Y_M \to X_M \xrightarrow{\pi} M \to 0$$

*such that $X_M$ is maximal Cohen–Macaulay and $Y_M$ has a resolution by $\omega_R^{\oplus}$ (i.e., $\mathsf{id}_R(Y_M) < \infty$). In particular, every homomorphism $f : X \to M$ with $X$ maximal Cohen–Macaulay factors through $\pi$.*

One of the main theorems of this talk is a generalization of this result to non-Cohen–Macaulay rings. To this end, we need to the notion of maximal Cohen–Macaulay complex.

**Definition 1.2.** Let $M^\bullet$ be an $R$-complex.

(1) The *depth* of $M^\bullet$ is defined by

$$\mathsf{depth}_R(M^\bullet) := \inf\{n \in \mathbb{Z} \mid \mathsf{Ext}_R^n(k, M^\bullet) \neq 0\}.$$

(2) The *dimension* of $M^\bullet$ is defined by

$$\mathsf{dim}_R(M^\bullet) := \sup\{\mathsf{dim}(R/\mathfrak{p}) + \sup(M_\mathfrak{p}^\bullet) \mid \mathfrak{p} \in \mathsf{Spec}(R)\}.$$

Here, $\sup(M^\bullet) = \sup\{n \in \mathbb{Z} \mid \mathsf{H}^n(M^\bullet) \neq 0\}$

(3) We say that $M^\bullet$ is a *maximal Cohen-Macaulay complex* if $M^\bullet \cong 0$ or $\mathsf{depth}_R(M^\bullet) = \mathsf{dim}_R(M^\bullet) = \mathsf{dim}(R)$.

**Remark 1.3.** In [2], the notion of a maximal Cohen–Macaulay complex has been introduced, though our definition is slightly weaker than theirs.

The following generalization of maximal Cohen–Macaulay approximation theorem is deduced from Auslander–Buchweitz theory on triangulated categories ([4]).

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**Theorem 1.4.** *Let $R$ be a noetherian local ring with dualizing complex $D_R$ (i.e., $R$ is a homomorphic image of a Gorenstein local ring). Let $M^\bullet$ be an $R$-complex with $\mathsf{dim}_R(M^\bullet) \leq \mathsf{dim}(R)$. Then there is an exact triangle*

$$Y_{M^\bullet} \to X_{M^\bullet} \xrightarrow{\pi} M^\bullet \to Y_{M^\bullet}[1]$$

*in the derived category such that $X_{M^\bullet}$ is a maximal Cohen–Macaulay complex and $Y_{M^\bullet}$ has a resolution by $D_R^{\oplus}$. In particular, every morphism $f : X^\bullet \to M^\bullet$ from a maximal Cohen–Macaulay complex $X^\bullet$ factors through $\pi$.*

## 2. Auslander–Reiten condition and Huneke–Jorgensen condition

In the study of Auslander–Reiten conjecture, a long-standing conjecture in commutative algebra, Auslander introduced the Auslander-Reiten condition. As an application of the first main theorem, we compare the Auslander-Reiten condition and its dual the Huneke-Jorgensen condition.

**Definition 2.1.** Let $M$ be a finitely generated $R$-module.

(1) We say that $M$ satisfies *Auslander-Reiten condition* ($\mathsf{ARC}$) if it satisfies:
there is an integer $n$ such that for any finitely generated $R$-module $N$,
$$\mathsf{Ext}_R^{\gg 0}(M,N) = 0 \Longrightarrow \mathsf{Ext}_R^{>n}(M,N) = 0.$$

(2) We say that $M$ satisfies *Huneke-Jorgensen consition* ($\mathsf{HJC}$) if it satisfies:
there is an integer $n$ such that for any finitely generated $R$-module $N$,
$$\mathsf{Tor}^R_{\gg 0}(M,N) = 0 \Longrightarrow \mathsf{Tor}^R_{>n}(M,N) = 0.$$

Huneke and Jorgensen ([3]) proved that these two conditions are in fact equivalent over a Gorenstein local ring.

**Theorem 2.2.** *If $R$ is a Gorenstein local ring, then every finitely generated $R$-module satisfies* ($\mathsf{ARC}$) *if and only if every finitely generated $R$-module satisfies* ($\mathsf{HJC}$).

The second main theorem extends this result in two directions:

**Theorem 2.3.** *Assume $R$ has a dualizing complex. Let $M$ be a finitely generated $R$-module.*

(1) *If $M$ satisfies* ($\mathsf{HJC}$)*, then $M$ satisfies* ($\mathsf{ARC}$).
(2) *Assume further that $R$ is Cohen-Macaulay or $\mathsf{Ext}_R^{\gg 0}(M,R) = 0$. Then $M$ satisfies* ($\mathsf{ARC}$) *if and only if $M$ satisfies* ($\mathsf{HJC}$).

## References


[1] M. Auslander and R.O. Buchweitz., The Homological theory of maximal Cohen–Macaulay approximations, *Societe Mathematique de France* **38** (1989), 5–37.

[2] S. B. Iyengar, L. Ma, K. Schwede, and M. E. Walker, Maximal Cohen–Macaulay complexes and their uses: A partial survey, *Commutative Algebra*, Springer, Cham, 475–500, 2021.

[3] C. Huneke and D. A. Jorgensen, Symmetry in the vanishing of Ext over Gorenstein rings, *Math. Scand.* **93** (2003), no. 2, 161–184.

[4] O. Mendoza, E. C. Sáenz, V. Santiago, and M. J. Souto Salorio, Auslander–Buchweitz approximation theory for triangulated categories, *Appl. Categ. Structures* **21** (2013), 119–139.